Inverse Problem for the Solid Cylinder

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For a solid cylindrical configuration, a temperature prediction method is developed based upon thermocouple data input from one location. Explicit formulations are presented for early times, as well as for unrestricted time spans. By the principle of temperature assessment at discrete locations, the resultant equations can be simplified for numerical computations. As demonstrated, high accuracy is feasible.

Nomenclature

a,b,c_n,D_n	= dimensionless coefficients in Eq. (21)
a_n	= dimensionless coefficients in Eq. (15)
A_n	= coefficient in Eq. (5)
$C_n(x)$	= dimensionless coefficients in Eq. (10)
$E(s)$	= correction to Eq. (18)
$F(s)$	= approximation function, Eq. (15)
$i^n \text{erfc}\, x$	$= \int_x^\infty i^{n-1}\text{erfc}\,\eta \,d\eta$
$I_0(x)$	= modified Bessel function of first kind of variable x
j,n	= summation index
J	= number of terms retained
m	= summation index, or constant
N	= degree of approximation, Eq. (5)
p	$= (s/\lambda)^{1/2}$
r	= radial position
r_0	= outer radius
s	= Laplace transform variable
t	= time
T	= temperature
$\bar{T}$	= Laplace transform of temperature
x	= dimensionless space variable r_1/r
x_1	= dimensionless spatial value r_1/r_0
$\bar{x}$	= dimensionless discrete location
λ	= thermal diffusivity
Δ	= arbitrary distance
δ	$=\Delta/r_1$
η_j	= dimensionless reduced time τ/a_j^2
ψ_j	= dimensionless reduced space value $(x_1^{-1} - x^{-1})/a_j$
τ	= dimensionless time $\lambda t/r_1^2$
Subscript	
1	= property at the thermocouple position

Introduction

RECENTLY, several analytical studies have appeared in the literature concerned with temperature prediction in heat-conducting solids.[1-3] In these methods, from an interior temperature trace coupled with either a boundary condition or another temperature trace, temperatures are predicted at other locations under certain restrictions. For planar geometries, the solutions are applicable for any time span so long as it is not greater than the time span of the input information, the thermocouple trace. For cylindrical geometries, the generation of an extrapolation mechanism is inherently more difficult, due to the presence of the modified Bessel functions. As shown in Refs. 4 and 5, limited short-time solutions are presented for the hollow cylindrical configuration. For the experimentally important situation involving solid cylindrical geometries, little information is available, although Ref. 1 touches upon this application briefly. Consequently, in what follows, an extrapolation procedure will be developed for the solid cylinder without the aforementioned restrictions on time. For completeness, a general short-time solution is presented, since it has utility in many engineering applications.

Received Feb. 28, 1978; revision received Sept. 13, 1978.
Index category: Heat Conduction.
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Short-Time Analysis

For radial heat flow in a solid cylinder, the mathematical representation of the inverse problem is

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] = \frac{1}{\lambda}\frac{\partial T}{\partial t} \tag{1}$$

where one interior condition is prescribed as

$$T(r_1,t) = T_1(t) \tag{2}$$

and thermal symmetry exists at the center:

$$\frac{\partial T}{\partial r} = 0 \text{ at } r=0 \tag{3}$$

By Laplace transform methods, Eq. (1) is transformed, and the transform temperature $\bar{T}(r,s)$, which satisfies the differential system, can be expressed as

$$\bar{T}(r,s) = \bar{T}_1(s) I_0(pr)/I_0(pr_1)$$

where

$$p = (s/\lambda)^{1/2} \tag{4}$$

At this point in the analysis, the temporal approximation of the temperature data at $r=r_1$ is selected to be the quasipower series:

$$T_1(t) = \sum_{n=1}^{N} A_n (4t)^n i^{2n}\text{erfc}\frac{m\Delta}{2(\lambda t)^{1/2}} \tag{5}$$

In reality, Eq. (5) is a simplified version of the temperature function $T_1(t)$ that appears in Ref. 2. For this application, summation over the index n is all that is required; hence the term $m\Delta$ represents collectively an effective length or distance to be specified. As will be demonstrated shortly, this approximation for the temperature trace greatly simplifies the analysis. Upon substitution into Eq. (4), the transform temperature may be rewritten as

$$\bar{T}(r,s) = \sum_{n=1}^{N} \frac{A_n n! \exp(-mp\Delta)}{s^{n+1}} \frac{I_0(pr)}{I_0(pr_1)} \tag{6}$$

where the quantity N represents the degree of the polynomial, Eq. (5), employed to approximate the temperature trace at $r=r_1$. For small or early times, the ratio of the modified Bessel functions in Eq. (6) is replaced with the appropriate asymptotic series expansions. After some algebraic manipulations, the ratio may be written as

$$\frac{I_0(pr)}{I_0(pr_1)} = \exp[p(r-r_1)]\left(\frac{r_1}{r}\right)^{1/2}\left\{1+\frac{r_1-r}{8pr_1r} + \frac{9r_1^2-7r^2-2r_1r}{128p^2r_1^2r^2} - \frac{1}{512}\left[\frac{37.5r_1^2-29.5r^3}{p^3r_1^3r^3} - \frac{9r_1+7r}{2p^3r_1^2r^2}\right]+\ldots\right\} \tag{7}$$

Substitution of the preceding equation into Eq. (6) produces

$$\bar{T}(r,s) = \sum_{n=1}^{N} \frac{A_n n! \exp[-p(m\Delta+r_1-r)]}{s^{n+1}}\left(\frac{r_1}{r}\right)^{1/2}\left\{1+\frac{r_1-r}{8pr_1r} + \frac{9r_1^2-7r^2-2r_1r}{128p^2r_1^2r^2} - \frac{1}{512}\left[\frac{37.5r_1^3-29.5r^3}{p^3r_1^3r^2} - \frac{9r_1+7r}{2p^3r_1^2r^2}\right]+\ldots\right\} \tag{8}$$

The value of the argument in the exponent now is selected so that the transform pairs will exist, i.e., $m\Delta+r_1-r\geq 0$. Accordingly, the temperature expression is obtained easily and may be written as

$$T(x,\tau) = x^{1/2}\sum_{j=0}^{J}\sum_{n=1}^{N} A_n n!\left(\frac{r_1^2}{\lambda}\right)^n C_j(x)(4\tau)^{2n+j/2} i^{2n+j} \times \operatorname{erfc}\frac{m\delta+1-x^{-1}}{2\tau^{1/2}}, \quad x\leq 1 \tag{9}$$

where dimensionless distance and time x, τ are defined, respectively, as $x=r_1/r$, $\tau=\lambda t/r_1^2$, and $\delta=\Delta/r_1$.

The quantity J indicates the number of terms retained in the step for the replacement of the Bessel function ratio, Eq. (7). For $J=4$, four terms are kept; consequently, the coefficients $C_j(x)$ associated with this approximation are

$$C_0 = 1 \tag{10a}$$

$$C_1(x) = x(x-1)/8 \tag{10b}$$

$$C_2(x) = 9x^2-2x-7/128 \tag{10c}$$

$$C_3(x) = -(75x^3-9x^2-7x-59)/1024 \tag{10d}$$

As previously noted, Eq. (9) is valid for any value of the term $m\delta$ so long as the relationship $m\delta\geq |1-x^{-1}|$ is satisfied. Two options exist at this point; the first is to select $m\delta$ so that it is a constant for the entire spatial range of interest. For convenience, if the cylinder's outer dimension is r_0, then an appropriate selection is $m\delta = |1-x_1^{-1}|$, where $x_1=r_1/r_0\leq 1$. Thus Eq. (9) determines the extrapolated temperatures throughout the region $x_1\leq x\leq 1$. A second alternative exists which can be especially fruitful if rapid assessment of the temperature is desired at discrete locations $x=\bar{x}$. Accordingly, $m\delta$ is chosen so that $m\delta = |1-(\bar{x})^{-1}|$; hence Eq. (9) is greatly simplified. In this instance, the expression for the temperature may be written alternatively as

$$T(\bar{x},\tau) = (\bar{x})^{1/2}\sum_{j=0}^{J}\sum_{n=1}^{N}\frac{A_n(\bar{x})n!(r_1^2/\lambda)^n C_j(\bar{x})\tau^{(2n+j)/2}}{\Gamma(2n+j+2)/2} \tag{11}$$

and the approximation of the temperature trace as

$$T_1(t) = \sum_{n=1}^{N} A_n(\bar{x})(4t)^n i^{2n}\operatorname{erfc}\frac{(\bar{x})^{-1}-1}{2\tau^{1/2}} \tag{12}$$

To emphasize the point that Eqs. (11) and (12) are designed for use at the discrete locations $x=\bar{x}$, the general coefficients A_n are rewritten as the term $A_n(\bar{x})$. Equation (11) determines the temperatures at the particular location $x=\bar{x}$ when the thermocouple output is replaced by its approximation, Eq. (12). Obviously, this approach has the added complication that, for different positions $\bar{x}$, the coefficients $A_n(\bar{x})$ must be recomputed, as against a single set of values as determined from Eq. (5); however, the predicted temperature formulation is considerably easier to evaluate. It should be noted that Eq. (12) is not recast entirely in dimensionless parameters, since the intent of the method is to utilize actual data rather than scaled-down information.

Complete Time Analysis

An expression for the predicted temperature for unrestricted time may be achieved in the following manner. Equation (6) is rewritten initially as

$$\bar{T}(r,s) = \sum_{n=1}^{N} A_n n! \exp[-p(m\Delta+r_1-r)]\frac{F(s)}{s^{n+1}} \tag{13}$$

when the function $F(s)$ is to be constructed so that it not only approximates closely the function involving the Bessel function ratio, i.e.,

$$F(s) \simeq \exp[-p(r-r_1)]I_0(pr)/I_0(pr_1) \tag{14}$$

but also consists of individual terms of which the transform pairs are readily available. Adopting the methodology presented in Ref. 1, $F(s)$ is assumed initially to be

$$F(s) = \frac{a_2pr_1}{1+a_3pr_1} + \frac{a_4}{1+a_6pr_1} \tag{15}$$

The coefficients a_n now are evaluated so that the resultant asymptotic expansions of Eq. (15) match the expansions of the right-hand side of Eq. (14) for early and long times. From this procedure, the following set of simultaneous equations is obtained:

$$a_2/a_3 = x^{1/2} \tag{16a}$$

$$a_4/a_6 - a_2/a_3^2 = (x-1)x^{1/2}/8 \tag{16b}$$

$$a_4 = 1 \tag{16c}$$

$$a_2 - a_4a_6 = -(1-x)/x \tag{16d}$$

The solution of Eqs. (16) establishes, in turn, the values for the terms a_n:

$$a_2 = a_3(x)^{1/2} \tag{17a}$$

$$a_4 = 1 \tag{17b}$$

$$a_6 = a_3(x)^{1/2} + (1-x)/x \tag{17c}$$

Table 1 Comparison of Eqs. (14) and (18)

pr_l	$\bar{x}=0.5$ Eq. (18)	Eq. (14)	% error	$x=0.6$ Eq. (18)	Eq. (14)	% error	$x=0.7$ Eq. (18)	Eq. (14)	% error	$x=0.8$ Eq. (18)	Eq. (14)	% error	$x=0.9$ Eq. (18)	Eq. (14)	% error
0.5	0.72793	0.72207	−0.812	0.79855	0.79591	−0.332	0.85934	0.85887	−0.055	0.91252	0.91285	0.036	0.95948	0.95946	0.002
1.0	0.66971	0.66238	−1.095	0.74780	0.73997	−1.058	0.81847	0.81250	−0.735	0.88344	0.87993	−0.397	0.94400	0.94235	−0.175
1.5	0.66040	0.66134	0.142	0.73663	0.73495	−0.229	0.80706	0.80545	−0.200	0.87342	0.87312	−0.034	0.93735	0.93798	0.067
2.0	0.66559	0.67098	0.804	0.73818	0.74213	0.465	0.80665	0.80997	0.410	0.87133	0.87533	0.459	0.93480	0.93862	0.407
3.0	0.68599	0.68583	−0.020	0.75112	0.75531	0.555	0.81491	0.82050	0.681	0.87538	0.88254	0.811	0.93500	0.94220	0.764
4.0	0.69350	0.69290	−0.086	0.76108	0.76184	0.100	0.82317	0.82606	0.350	0.88129	0.88666	0.609	0.93759	0.94445	0.726
5.0	0.69923	0.69646	−0.395	0.76677	0.76513	−0.214	0.82857	0.82885	0.034	0.88590	0.88875	0.321	0.94038	0.94561	0.553
7.0	0.70321	0.69999	−0.460	0.77106	0.76830	−0.359	0.83328	0.83150	−0.214	0.89083	0.89071	−0.014	0.94447	0.94669	0.235
∞	0.70711	0.70711	0	0.77459	0.77459	0	0.83666	0.83666	0	0.89443	0.89443	0	0.94869	0.94869	0

Table 2 Coefficients for Eq. (21)

	$\bar{x}=0.7$	$\bar{x}=0.8$	$\bar{x}=0.9$
D_0	−0.0543	0.00675	0.005
D_1	0.00073	−0.00924	−0.01572
D_3	0.35519	0.08452	0.08173
c_0	2.20569	0.57723	0.33606
c_1	0.15004	0.92482	1.08338
c_2	4.00291	2.34650	2.18275
a	6	2	2
b	0.75	0.75	1

$$a_3 = -\left\{\frac{8x^{1/2}+1-x}{2x^{3/2}}\right\}\left\{1-\left[1+\frac{32x^{3/2}}{(1-x+8x^{1/2})^2}\right]^{1/2}\right\},\quad x\neq 1 \tag{17d}$$

For the limiting condition of $x=1$, $a_2=a_3=a_6$, the approximating function is identically equal to unity, which is in accordance with Eq. (14). By construction, the initial form of the function $F(s)$ produces accurate results near the limits; however, it can be anticipated that Eq. (15) will have to be amended for other values. Based upon a numerical comparison of Eqs. (14) and (15) for midrange values of the parameters pr_l, a parametric correction is developed. Thus, a more accurate determination for $F(s)$ can be established. For values of $x\geq 0.5$, good agreement is achieved when the function $F(s)$ is altered to

$$F(s)=\frac{a_2pr_l}{1+a_3pr_l}+\frac{a_4}{1+a_6pr_l}-0.46379pr_l(1-x)^{1.25}\times\exp[-1.39028pr_l(1-x)^{0.4}] \tag{18}$$

As shown in Table 1, the function $F(s)$ approximates the right-hand side of Eq. (14) quite well for a realistic range in the spatial variable $x\geq 0.5$. The tabulated results indicate that the difference is not more than 1.1% in error, and this occurs, an occasion, at an isolated value for the parameter pr_l. In many instances, the error is much less and for all practical purposes may be disregarded. Proceeding with the analysis, substitution of Eq. (18) into Eq. (13) establishes the relationship

$$\bar{T}(x,s)=\sum_{n=1}^{N}\frac{A_n n!x^{1/2}}{s^{n+1}}\exp[-pr_l(m\delta+1-x^{-1})]\left\{\frac{a_2pr_l}{1+a_3pr_l}+\frac{1}{1+a_4pr_l}-0.46379pr_l(1-x)^{1.25}\times\exp[-1.39208pr_l(1-x)^{0.4}]\right\} \tag{19}$$

As in the development of the early time solution, two options exist for suppression of the argument of the exponent. The first is to select $m\delta=|1-x_l^{-1}|$, where $x_l\leq 1$. Since the transform pairs exist for each of the terms in braces, the inverse of the transform temperature is, therefore,

$$T(x,\tau)=\sum_{j=3,6}(-1)^{j+1}x^{3/j}\sum_{n=1}^{N}A_n n! a_j^{2n}\left(\frac{r_l^2}{\lambda}\right)^n\left\{\exp(\eta_j+\psi_j)\times\mathrm{erfc}\left(\frac{\eta_j}{2\psi_j^{1/2}}+\psi_j^{1/2}\right)-\sum_{m=0}^{6(n-1)+j/3}(-2\psi_j^{1/2})^m i^m\mathrm{erfc}\left(\frac{\eta_j}{2\psi_j^{1/2}}\right)\right\}-0.46379\sum_{n=1}^{N}A_n n!x^{1/2}(1-x)^{1.25}\left(\frac{r_l^2}{\lambda}\right)^n\left\{(4\tau)^{2n-1/2}i^{2n-1}\times\mathrm{erfc}\left(\frac{x_l^{-1}-x^{-1}+1.39208(1-x)^{0.4}}{2\tau^{1/2}}\right)\right\} \tag{20}$$

where $\psi_j=\tau/a_j^2$ and $\eta_j=(x_l^{-1}-x^{-1})/a_j$.

Equation (20) predicts the temperature throughout the range $x_l\leq x\leq 1$ in dimensional time and space variables τ and x, respectively. For discrete temperature assessment, $m\delta$ is chosen so that the relationship $m\delta=|1-(\bar{x})^{-1}|$ is satisfied. In this instance, the equation for the transform temperature is obtained from Eq. (19) by removing the exponential contribution and setting $x=\bar{x}$. Since the desired location now is fixed, further improvements in the approximation function, Eq. (18), can be introduced. This correction, $E(s)$, is expressed as

$$E(s)=D_0pr_l\exp(-c_0pr_l)+D_1(a-pr_l)pr_l\exp(-c_1pr_l)+D_2(b-pr_l)pr_l\exp(-c_2pr_l) \tag{21}$$

In Table 2, the numerical values of the constants a,b,D_n, and c_n are shown for several different values of the predetermined spatial value $x=\bar{x}$. Consequently, a new approximation for the function $F(s)$ is obtained by the addition of the right-hand sides of Eqs. (18) and (21). As a demonstration of the accuracy that can be achieved, numerical results are presented in Table 3. The error is less than 0.3% in all instances. For temperature extrapolation to discrete locations, $x=\bar{x}$, the inverse of Eq. (19), as amended, is

$$T(\bar{x},\tau)=\sum_{n=1}^{N}A_n n!\bar{x}^{1/2}\left(\frac{r_l^2}{\lambda}\right)^n\left\{\bar{x}^{1/2}a_3^{2n}\left[\exp(\psi_3)\mathrm{erfc}\psi_3^{1/2}-\sum_{m=0}^{2n-1}\frac{(-\psi_3)^m}{\Gamma(m+2/2)}\right]-a_6^{2n}\left[\exp(\psi_6)\mathrm{erfc}\psi_6^{1/2}-\sum_{m=0}^{2n}\frac{(-\psi_6)^m}{\Gamma(m+2/2)}\right]\right\}-0.46379\sum_{n=1}^{N}A_n n!\bar{x}^{1/2}(1-\bar{x})^{1.25}\times\left(\frac{r_l^2}{\lambda}\right)^n\left\{(4\tau)^{2n-1/2}i^{2n}\mathrm{erfc}\frac{1.39208(1-\bar{x})^{0.4}}{2\tau^{1/2}}\right\}$$

Table 3 Comparison of Eq. (14) and amended Eq. (18)

	$\bar{x}=0.9$		$\bar{x}=0.8$		$\bar{x}=0.7$	
pr_1	Eqs. (18) and (21)	% error	Eqs. (18) and (21)	% error	Eqs. (18) and (21)	% error
0.5	0.96159	−0.222	0.91396	−0.122	0.85800	0.101
1.0	0.94225	0.010	0.88156	−0.185	0.81388	−0.170
1.5	0.93724	0.079	0.87195	0.134	0.80697	−0.188
2.0	0.93783	0.084	0.87365	0.192	0.80933	0.078
3.0	0.94160	0.064	0.88017	0.268	0.81886	0.200
4.0	0.94430	0.016	0.88568	0.110	0.82634	−0.034
5.0	0.94606	−0.047	0.88910	−0.040	0.83029	−0.174
7.0	0.94808	−0.147	0.89215	−0.161	0.83149	0.001
∞	0.94869	0	0.89443	0	0.83666	0

$$+\sum_{n=1}^{N} A_n n! \bar{x}^{1/2} \left(\frac{r_1^2}{\lambda}\right)^n \Big\{ (4\tau)^{2n-1/2} \Big[D_0 i^{2n-1}\operatorname{erfc}\frac{c_0}{2\tau^{1/2}} + aD_1 i^{2n-1}\operatorname{erfc}\frac{c_1}{2\tau^{1/2}} + bD_2 i^{2n}\operatorname{erfc}\frac{c_2}{2\tau^{1/2}} \Big] - (4\tau)^{n-1}\Big[D_1 i^{2(n-1)}\operatorname{erfc}\frac{c_1}{2\tau^{1/2}} + D_2 i^{2(n-1)}\operatorname{erfc}\frac{c_2}{2\tau^{1/2}} \Big] \Big\} \quad (22)$$

when Eq. (12) is used in conjunction with the preceding equation. The last summation in the preceding equation is the contribution of the correction $E(s)$. For many applications, its presence can be eliminated; hence the resultant formulation can be evaluated easily.

Summary

Based upon input from one thermocouple sensor, explict temperature extrapolation expressions are developed for short or early times in a solid cylindrical configuration. As shown in the theoretical development, a rather simple expression is feasible if temperature assessment is desired at predetermined locations. In many practical applications, the entire temperature field may not be required; consequently this approach is beneficial. For unrestricted time situations, additional formulations are presented. As demonstrated, an approximating function $F(s)$ is generated parametrically. The resultant accuracy should prove to be adequate for many engineering applications. As in the short-time solution, an alternative expression is derived for discrete spatial locations. Since the location is now fixed, a correction to Eq. (18) can be determined easily. Fortuitously, this amendment may be represented by the single equation, Eq. (21), and a tabulation of the related coefficients is presented in Table 2 for a practical range in the discrete locations $\bar{x}$. In either situation, it should be noted that the internal temperature trace $T_1(t)$ must be responding dynamically; consequently the coefficients A_n, as determined from Eq. (5), should be realistic values. In instances where this may not be the case, i.e., high surface flux coupled with small or short time, it is always possible to relocate the one interior thermocouple so that it is closer to the surface and thereby produces a sensor trace that is more representative of the heating mode. For well-designed experimental situations, i.e., judicious thermocouple placement, several sensors customarily are employed; consequently the preceding instance presents no difficulty. Furthermore, if an assessment of the surface flux is desired, obviously Eqs. (9) and (19) may be differentiated for the appropriate quantities. In conclusion, a method of temperature prediction is developed based upon temperature input from one thermocouple rather than from two sensors. For applications where one of the two thermocouples is not responding sufficiently, the current theoretical results may be employed satisfactorily.

References

[1] Sparrow, E. M., Hadji-sheikh, A., and Lundgren, T.S., "The Inverse Problem in Transient Heat Conduction," *Journal of Applied Mechanics*, Vol. 86, 1964, pp. 369-375.

[2] Imber, M. and Khan, J., "Prediction of Transient Temperature Distributions with Embedded Thermocouples," *AIAA Journal*, Vol. 10, June 1972, pp. 784-789.

[3] Mehta, R. C., "Solution of the Inverse Conduction Problem," *AIAA Journal*, Vol. 15, Sept. 1977, pp. 1355-1358.

[4] Imber, M., "A Temperature Extrapolation Method for Hollow Cylinders," *AIAA Journal*, Vol. 11, Jan. 1973, pp. 117-118.

[5] Chen, C. J. and Thomsen, D. M., "On Transient Cylindrical Surface Heat Flux Predicted from Interior Temperature Response," *AIAA Journal*, Vol. 13, May 1975, pp. 697-699.